INTRODUCTION TO BIOMECHANICS
317.043, VU

Point Mechanics

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Outline

- Kinematics
- Kinetics
- Equilibrium
- Power and Energy

Introduction to Biomechanics – Point Mechanics - Learning Outcomes

Aims

- Point mechanics refresher
- Kinematics, Kinetics, Equilibrium, Power and Energy
- Mechanics elements – representation

Planned learning outcomes

- Ability to describe and explain:
  - The mechanics of a mass concentrated in a point
  - Newton's laws
- Ability to solve simple point mechanics problems

Idealisations

- Particle
  - Has mass, size can be neglected – geometry not taken into account (a matter of relative size)
- Rigid body
  - Mass and size, consider large collective of particles
- Concentrated force
  - Effect of loading assume to act at single point on a body

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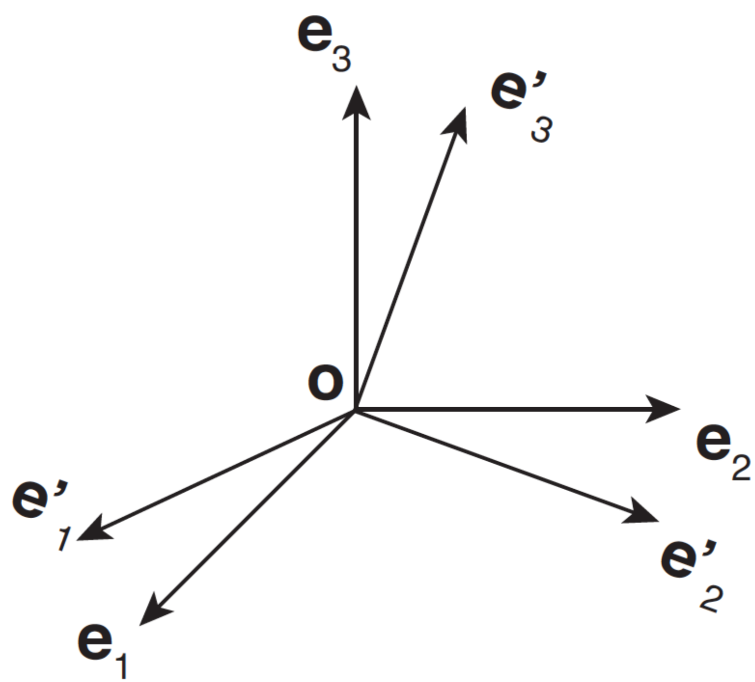
Kinematics

Position of the point

$$\mathbf{R}(t) = R_{ij}(t)\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)$$

$$\mathbf{x}(t) = x_i(t)\mathbf{e}_i$$



In the rotated frame:

$$\mathbf{x}(t) = x'_j(t)\mathbf{e}'_j$$
$$= x'_j(t)\mathbf{R}(t)\mathbf{e}_i$$

$$x_i(t) = R^T_{ij}x'_j(t)$$

$$x'_i(t) = R_{ij}x_j(t)$$

Velocity and acceleration

Velocity is the (time-) derivative of position:

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} \qquad v_i = \frac{dx_i}{dt}$$

Acceleration is the time derivative of velocity:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\left(\|\mathbf{v}\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)}{dt} = \underbrace{\frac{d\|\mathbf{v}\|}{dt}\frac{\mathbf{v}}{\|\mathbf{v}\|}}_{\text{tangential component}} + \underbrace{\|\mathbf{v}\|^2 \frac{d\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)}{ds}}_{\text{normal component}}$$

tangential component

normal component

Generalized coordinates

Parametrization of every position by real numbers *(q_1 , q_2 , q_3 , t)*, such that

$$x_1 = x_1(q_1, q_2, q_3, t)$$

$$x_2 = x_2(q_1, q_2, q_3, t)$$

$$x_3 = x_3(q_1, q_2, q_3, t)$$

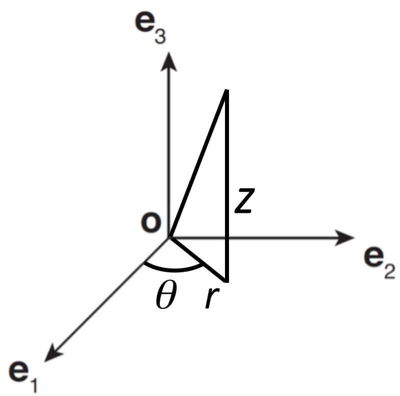
Velocity

Velocity is time-derivative of position

$$v_i = \frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial q_j}\dot{q}_j$$

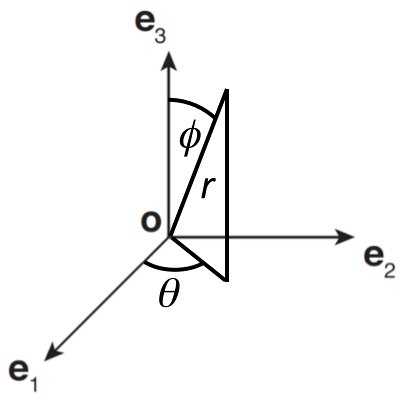
In cylindrical coordinates:

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$$



In spherical coordinates:

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\sin(\theta)\dot{\phi}\mathbf{e}_\phi$$



Acceleration

Acceleration is time-derivative of velocity

$$a_i = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial q_j} \dot{q}_j$$

In cylindrical coordinates:

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\mathbf{e}_z$$

In spherical coordinates:

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\sin^2(\theta)\dot{\phi}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin(\theta)\cos(\theta)\dot{\phi}^2\right)\mathbf{e}_\theta$$

$$+\left(r\sin(\theta)\ddot{\phi} + 2\sin(\theta)\dot{r}\dot{\phi} + 2r\cos(\theta)\dot{\theta}\dot{\phi}\right)\mathbf{e}_\phi$$

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Mass

Key properties:

- Scalar
- Positive $m \geq 0$
- Conserved ?

Linear momentum

The linear momentum is the product of mass with velocity:

$$\mathbf{p} = m\mathbf{v}$$

with units *kg m/s*

When mass is conserved: $\mathbf{p} = \dfrac{d\left(m\mathbf{r}\right)}{dt}$

Angular momentum

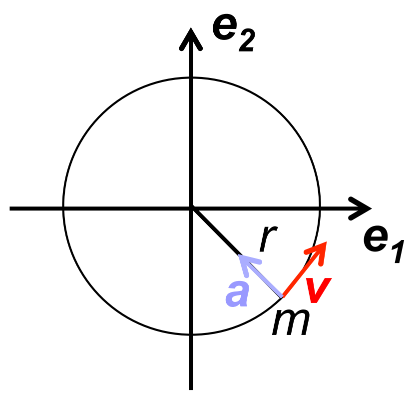
Consider a particle with mass *m* on a circular trajectory with constant speed *v:*

$$\mathbf{r} = r\mathbf{e}_r(t) \qquad \theta = \omega t$$

From cylindrical coordinates we know:

$$\mathbf{v} = r\omega\mathbf{e}_\theta \qquad \mathbf{a} = -r\omega^2\mathbf{e}_r$$

$$\mathbf{p} = m\mathbf{v} = mr\omega\mathbf{e}_\theta$$



Angular momentum:

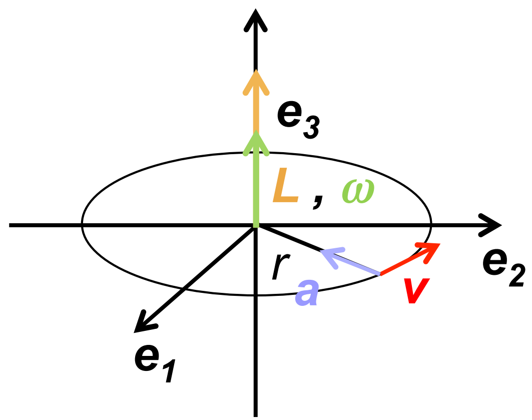
$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = (r)\mathbf{e}_r \times (mr\omega)\mathbf{e}_\theta = (mr^2\omega)\mathbf{e}_z$$

$$\mathbf{L} = \mathbf{I}(\omega\mathbf{e}_z)$$

Angular momentum

I is the moment of inertia (generally a second rank tensor) in the case of a point mass on circular trajectory it reduces to:

$$I = mr^2$$



Generally we can also write:

$$\dot{\mathbf{r}} = \underline{\omega} \times \mathbf{r}$$

As:

$$\underline{\omega} \times \mathbf{r} = \omega \mathbf{e}_z \times r\mathbf{e}_r = \omega r \left(\mathbf{e}_z \times \mathbf{e}_r \right) = \omega r \mathbf{e}_\theta$$

The pseudo-vectors ***L*** and $\boldsymbol{\omega}$ point in x_3 direction

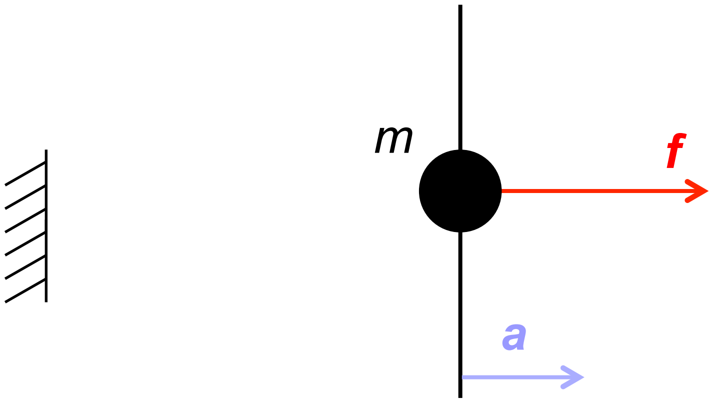
Forces

External action on a system which leads to a change of the original state or a change of motion

Characteristics:

- Intensity
- Direction
- Sign

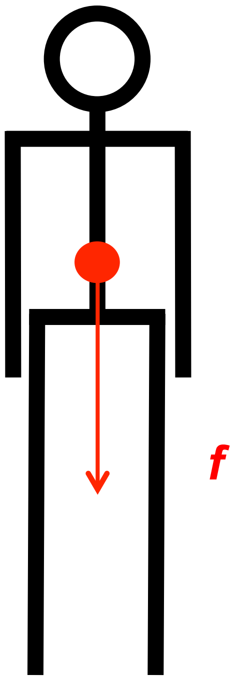
Forces: inertia



$$\mathbf{f} = m\mathbf{a} = m\ddot{\mathbf{x}}$$

Inertia: the higher the mass the lower acceleration due to application of a given force

Forces: gravity



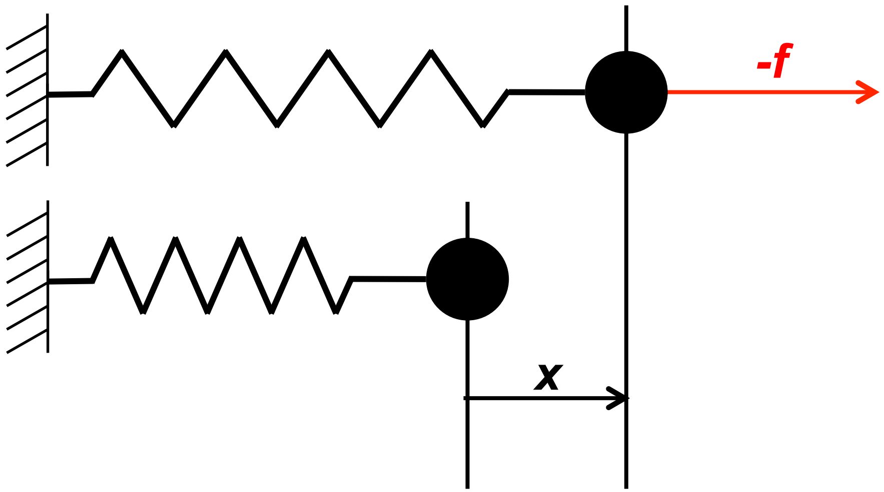
$$\mathbf{f} = m\mathbf{g}$$

Force proportional to acceleration of gravity

Forces: rheological elements

- Forces acting on materials can lead to different behavior
- Dependent on material properties
- Elastic or inelastic
  - Elastic – full recoil, resilient
  - Inelastic – does not reach initial configuration after unloading
- Can use rheological element or combination to describe overall behaviour

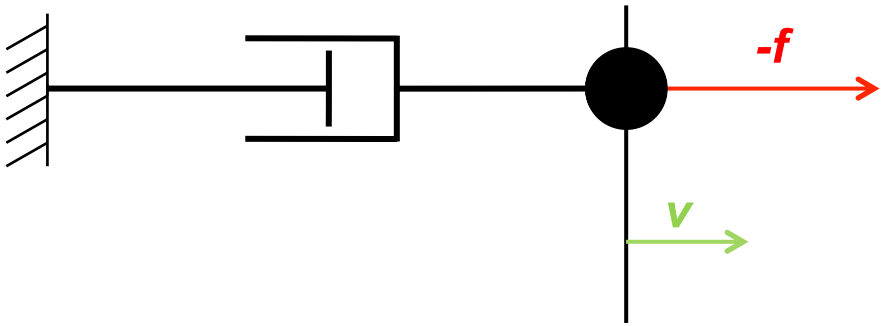
Forces - rheological elements: spring



$$\mathbf{f} = -k\mathbf{x}$$

Force varies linearly with elongation – linear elastic element

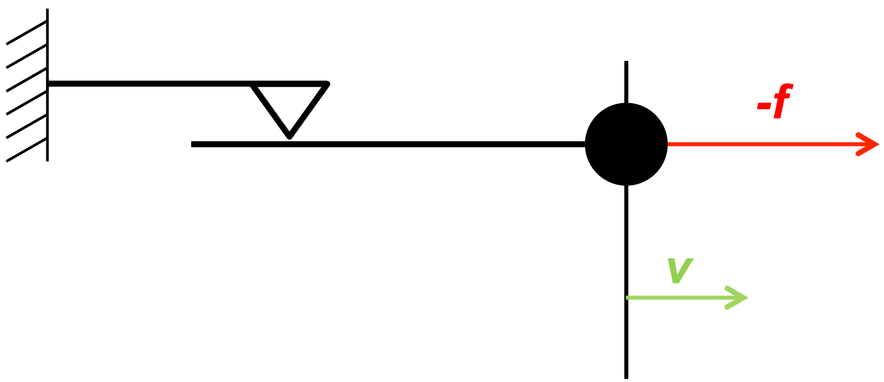
Forces - rheological elements: dashpot / damper



$$\mathbf{f} = -D\dot{\mathbf{x}}$$

Force varies linearly with velocity – viscoelastic (inelastic) element

Forces - rheological elements: slider



$$\mathbf{f} = -\mu \frac{\dot{\mathbf{x}}}{\|\dot{\mathbf{x}}\|}$$

Force is constant upon presence of force and antagonistic – plastic (inelastic) element

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Newton's laws of mechanics

First law – body remain at rest or continue uniform motion on straight line unless external force present

$$\frac{d\mathbf{p}}{dt} = 0 \Leftrightarrow \sum_i {}^{i}\mathbf{f} = 0$$

Second law – external force causes acceleration

$$\frac{d\mathbf{p}}{dt} = \sum_i {}^{i}\mathbf{f}$$

Third law – actio = reactio

$${}^{actio}\mathbf{f} = -{}^{ji}\mathbf{f}$$

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Work

A force does work when it undergoes a displacement in the direction of its line of action:

$$W = \mathbf{f} \cdot \mathbf{x} \qquad\qquad dW = \mathbf{f} \cdot d\mathbf{x}$$

Power is the rate of work done per unit time:

$$P = \frac{dW}{dt} = \mathbf{f}\frac{d\mathbf{x}}{dt} = \mathbf{f} \cdot \mathbf{v}$$

Or in other words: this is the power developed by a force bound to a position ***x***

Power and work

With Power being:

$$P = \mathbf{f} \cdot \mathbf{v} = \mathbf{f} \cdot \dot{\mathbf{x}} = m\mathbf{a} \cdot \mathbf{v} = m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}^2\right)$$

Force is the conjugate variable of velocity for power.

Work done by external forces in interval $[t_0, t]$ is

$$\Delta W = \int_{t_0}^{t} P(\tau)d\tau$$

Potential energy

Conservative forces are defined by

$$\oint \mathbf{f}\, d\mathbf{x} = 0$$

This means a potential function *U* exists such that

$$\mathbf{f}(\mathbf{x}) = -\nabla U(\mathbf{x})$$

The force ***f*** derives from a potential *U*, which contributes to the potential energy:

$${}^{pot}E(\mathbf{x}) = \sum_{i} {}^{i}U(\mathbf{x})$$

Potential energy

Consider case of spring element – spring oriented parallel to *x*-direction (x_1):

$$f(x) = -kx$$

$$\oint f\,dx = \oint -kx\,dx = 0$$

The force derived from the potential

$$^{pot}E(x) = \frac{1}{2}kx^2$$

Potential energy

Consider gravity

$$\mathbf{f} = m\mathbf{g}$$

$$\oint m\mathbf{g}\, d\mathbf{x} = 0$$

The force derived from the potential

$${}^{pot}E\left(\mathbf{x}\right) = -m\mathbf{g} \cdot \mathbf{x}$$

Energy

The kinetic energy is defined as

$$ {}^{kin}E(x) = \frac{1}{2} m \|\mathbf{v}\|^2 $$

The theorem of the kinetic energy is

$$ \frac{d\,{}^{kin}E}{dt} = -\frac{d}{dt}\sum_i {}^{i}U(\mathbf{x}) + {}^{nc}P $$

Where ${}^{nc}P$ is the power dissipated by non-conservative forces

Mechanical energy

Mechanical energy is the sum of kinetic and potential energy

$$^{mec}E = {}^{kin}E + {}^{pot}E$$

The theorem of mechanical energy hence denotes

$$\frac{d\,^{mec}E}{dt} = {}^{nc}P$$

Conservation of energy

Energy is conserved if all forces are conservative

$$\frac{d\,{}^{mec}E}{dt} = {}^{nc}P = 0 \qquad {}^{mec}E(t) = {}^{mec}E(t_0) \quad \forall t$$

This means that variations of kinetic and potential energy are opposed

$$\frac{d\,{}^{mec}E}{dt} = -\frac{d\,{}^{pot}E}{dt}$$

Dissipation of energy

Energy is not conserved if a force is not conservative, e.g. dashpot / damper

$$\oint f\,dx = \oint -D\dot{x}\,dx = \oint -D\|\dot{x}\|^2\,d\tau = 0 \Leftrightarrow \dot{x} \equiv 0$$

$$^{nc}P = D\|\dot{x}\|^2 \geq 0$$

or slider

$$\oint f\,dx = \oint -\mu\frac{\dot{x}}{\|\dot{x}\|}dx = \oint -\mu\|\dot{x}\|\,d\tau = 0 \Leftrightarrow \dot{x} \equiv 0$$

$$^{nc}P = \mu\|\dot{x}\| \geq 0$$

Galilean relativity principle

The forces and the laws of equilibrium are invariant with respect to Galilean transformation, i.e. change of reference system that is uniform motion with respect to the original one.

So the forces and laws will be equivalent for two observers in uniform motion with respect to each other.

Summary

- Kinematics – position, velocity, acceleration
- Kinetics – mass, linear, angular momentum, forces
- Equilibrium – Newton's laws
- Power and energy – work, conservative vs. dissipative forces, potential energy