1. ① $\ddot{z} + \gamma \dot{z} + \omega^2 z = F = k a \cos(\Omega t)$

② ansatz:

$$z_p = A\cos(\Omega t) + B\sin(\Omega t)$$

$$\dot{z}_p = -\Omega A \sin(\Omega t) + B\Omega \cos(\Omega t)$$

$$\ddot{z}_p = -\Omega^2 A\cos(\Omega t) - B\Omega^2 \sin(\Omega t)$$

$$\Rightarrow \ddot{z} + \gamma \dot{z} + \omega^2 z = A(\omega^2 - \Omega^2)\cos\Omega t + B(\omega^2 - \Omega^2)\sin\Omega t$$

$$+ \gamma B\Omega \cos\Omega t - \gamma A\Omega \sin\Omega t$$

$$= k a \cos(\Omega t)$$

$$\Rightarrow \begin{cases} A(\omega^2 - \Omega^2) + B\gamma\Omega - ka = 0 & \leftarrow \text{The coefficient of } \cos(\Omega t) \\ B(\omega^2 - \Omega^2) - A\gamma\Omega = 0 & \leftarrow \text{The coefficient of } \sin\Omega t; \end{cases}$$

$$\Rightarrow \begin{cases} A = \dfrac{K(\omega^2 - \Omega^2)}{(\omega^2 - \Omega^2)^2 + \gamma^2\Omega^2} \\ B = \dfrac{K\gamma\Omega}{(\omega^2 - \Omega^2)^2 + \gamma^2\Omega^2} \end{cases}$$

The catastrophe means the denominator is in its minimum value!

So: $$\frac{d\left((\omega^2-\Omega^2)^2+\gamma^2\Omega^2\right)}{d\Omega^2}=0$$

$$\Rightarrow \quad 2(\Omega^2-\omega^2)+\gamma^2=0$$

$$\Rightarrow \quad \Omega^2=\omega^2-\frac{\gamma^2}{2} \Rightarrow \Omega=\sqrt{\omega^2-\gamma^2/2}$$

③ $$z_p(t)=A\cos\Omega t+B\sin\Omega t$$

$$=\sqrt{A^2+B^2}\left(\frac{A\cos(\Omega t)}{\sqrt{A^2+B^2}}+\frac{B\sin(\Omega t)}{\sqrt{A^2+B^2}}\right)$$

let $\frac{A}{\sqrt{A^2+B^2}}=\cos\varphi$, $\frac{B}{\sqrt{A^2+B^2}}=\sin\varphi$

$$z_p=\sqrt{A^2+B^2}\,(\cos\Omega t\cos\varphi-\sin\Omega t\sin\varphi)$$

$$=\sqrt{A^2+B^2}\,\cos(\Omega t-\varphi)$$

$$=C\cos(\Omega t-\varphi)$$

$$\Rightarrow \quad C=\sqrt{A^2+B^2}=\frac{k_A}{\sqrt{(\omega^2-\Omega^2)^2+\gamma^2\Omega^2}}$$

$$\tan\varphi=\frac{B}{A}=\frac{\gamma\Omega}{(\omega^2-\Omega^2)}$$

let: $z=z_p+z_h$

$$\Rightarrow \dot{z}=\dot{z}_p+\dot{z}_h$$

$$\ddot{z}=\ddot{z}_p+\ddot{z}_h$$

$$\Rightarrow \ddot{z} + \gamma \dot{z} + \omega^2 z = \cancel{\theta}$$

$$= (\ddot{z}_p + \gamma \dot{z}_p + \omega^2 z_p) + (\ddot{z}_h + \gamma \dot{z}_h + \omega^2 z_h)$$

$$= F + 0 = \underline{\underline{F}}$$

recall $z_h = a e^{-\frac{\gamma}{2}t} \cos(\omega' t + \delta)$, where $\omega' = \sqrt{\omega^2 - \frac{\gamma^2}{4}}$

when $\gamma^2 < 4\omega^2$.

$$z_p = C \cos(\Omega t - \varphi)$$

So ~~z~~ the general solution of z is:

$$z = a e^{-\frac{\gamma}{2}t} \cos(\omega' t + \delta) + C \cos(\Omega t - \varphi).$$

when $t \to \infty$

$$\underline{\underline{z = C \cos(\Omega t - \varphi)}}$$

i.e. the friction will disappear!!

2. ① $\vec{F} = m\vec{a}$

$$\Rightarrow \begin{cases} F_x = ma_x \Rightarrow m\ddot{x} = -k_x x \Rightarrow m\ddot{x} + k_x x = 0 \\ F_y = ma_y \Rightarrow m\ddot{y} = -k_y y \Rightarrow m\ddot{y} + k_y y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{x} + \frac{k_x}{m} x = 0 \\ \ddot{y} + \frac{k_y}{m} y = 0 \end{cases} \Rightarrow \begin{cases} x = A_x \cos\sqrt{\frac{k_x}{m}}\,t + B_x \sin\sqrt{\frac{k_x}{m}}\,t \\ y = A_y \cos\sqrt{\frac{k_y}{m}}\,t + B_y \sin\sqrt{\frac{k_y}{m}}\,t. \end{cases}$$

② let the potential be $u(x,y)$

$$\Rightarrow \begin{cases} F_x = -\frac{\partial u}{\partial x} \Rightarrow -\frac{\partial u}{\partial x} = -k_x x \quad ① \\ F_y = -\frac{\partial u}{\partial y} \Rightarrow -\frac{\partial u}{\partial y} = -k_y y \quad ② \end{cases}$$

From ①, we get

$$u = \frac{1}{2} k_x x^2 + f(y) \quad ③$$

plug ③ into ②, we get

$$f'(y) = k_y y \Rightarrow f(y) = \frac{1}{2} k_y y^2 + C$$

So: $u(x,y) = \frac{1}{2}(k_x x^2 + k_y y^2) + \underset{0}{C}$

The total energy:

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 + u(x,y)$$

$$= \frac{1}{2}(k_x x^2 + k_y y^2) + \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

③ $L_z = x P_y - y P_x = x m \dot{y} - y m \dot{x}$

$\dot{L} = \cancel{m\dot{x}\dot{y}} + m x \ddot{y} - \cancel{m\dot{x}\dot{y}} - m \ddot{x} y$

$= m x \ddot{y} - m \ddot{x} y$

Recall the equation of motion:

$$\begin{cases} m\ddot{x} = -k_x x = -kx \\ m\ddot{y} = -k_y y = -ky \end{cases}$$

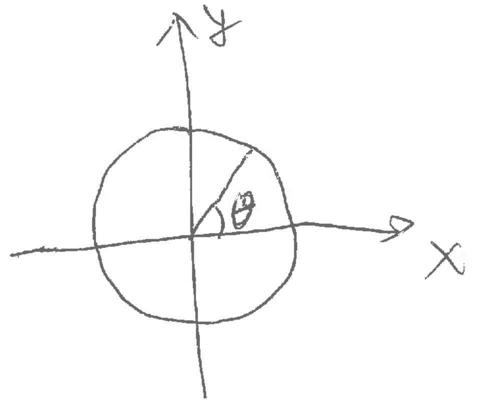
So $\dot{L} = -kxy + kxy = \underline{\underline{0}}$

3. ① $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -\frac{By}{r^2} & \frac{Bx}{r^2} \end{vmatrix} = B\left(\frac{\partial\left(\frac{x}{r^2}\right)}{\partial x} + \frac{\partial\left(\frac{y}{r^2}\right)}{\partial y} \right)$

$= \left(\frac{1}{r^2} - \frac{2x^2}{r^4} + \frac{1}{r^2} - \frac{2y^2}{r^4} \right)$

$= \frac{2}{r^2} - \frac{2(r^2)}{r^4} = \underline{\underline{0}}$

②



The range of θ goes from 0, to 2π.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad d\vec{r} = (r\,d\cos\theta,\ r\,d\sin\theta) = r(-\sin\theta,\ \cos\theta)\,d\theta$$

$$\vec{F} = \left(\frac{-Br\sin\theta}{r^2},\ \frac{Br\cos\theta}{r^2} \right) = \frac{B}{r}(-\sin\theta,\ \cos\theta)$$

$$\vec{F}\cdot d\vec{r} = B(\sin^2\theta + \cos^2\theta)\,d\theta = 1\,d\theta$$

$$\int_0^{2\pi} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} B\,d\theta = 2\pi B$$

③ $u = -B\arctan\frac{y}{x}$

$$\Rightarrow \begin{cases} -\dfrac{\partial u}{\partial x} = B\dfrac{\left(-\frac{y}{x^2}\right)}{1+\frac{y^2}{x^2}} = \dfrac{-By}{x^2+y^2} = -\dfrac{By}{r^2} \\[2ex] -\dfrac{\partial u}{\partial y} = B\left(\dfrac{\frac{1}{x}}{1+\frac{y^2}{x^2}}\right) = B\dfrac{x}{x^2+y^2} = B\dfrac{x}{r^2} \end{cases}$$

④ $\vec{\nabla} \times \vec{F} = 0$ should be satisfied at every point!

However, $\vec{F}$ is not well-defined at $r=0$!

~~also~~ nor u!

4. One dimensional harmonic oscillator:

① The canonical momentum is

$$p = mv = m\dot{z}$$

The kinematic energy $= \frac{1}{2} m \dot{z}^2 = \frac{p^2}{2m}$

potential energy $= \frac{1}{2} k z^2$

The total energy: $E = \frac{p^2}{2m} + \frac{1}{2} k z^2$

So $\Rightarrow \frac{p^2}{2mE} + \frac{k z^2}{2E} = 1$

$\Rightarrow \frac{p^2}{2mE} + \frac{z^2}{\left(\frac{2E}{k}\right)} = 1$

$\Rightarrow a = \sqrt{2mE} \quad b = \sqrt{\frac{2E}{k}}$

③ The trajectory on p-z plane is ellipse.

From $\begin{cases} a = \sqrt{2mE} \Rightarrow E = \frac{a^2}{2m} \\ b = \sqrt{\frac{2E}{k}} \Rightarrow E = b^2 k/2 \end{cases}$

Recall: $E = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} k z^2$

$$\frac{dE}{dt} = m \ddot{z} \dot{z} + k z \dot{z} = \underbrace{(m \ddot{z} + k z)}_{= 0} \dot{z} = 0$$

Equation of motion!